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Conquest of the ghost pyramid of the superstring

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Abstract: We give a new Becchi-Rouet-Stora-Tyutin operator for the superstring. It implies a quadratic gauge-fixed action, and a new gauge-invariant action with first-class constraints. The infinite pyramid of spinor ghosts appears in a simple way through ghost gamma matrices.

Keywords: BRST Quantization, BRST Symmetry, Superstrings and Heterotic Strings, Superspaces.

Contents

1. Introduction

The advantages of supersymmetry are somewhat obscured in the Ramond-Neveu-Schwarz formalism, as is the case for supersymmetric particle theories when not formulated in superspace. For example, cancellations of divergences are not obvious, and amplitudes with many fermions are difficult to calculate.

Some of these problems were resolved with the Green-Schwarz formalism, but it proved difficult to quantize except in the lightcone gauge, where some manifest supersymmetry is retained in trade for the loss of some manifest Lorentz invariance. (Similar remarks apply to the Casalbuoni-Brink-Schwarz superparticle.) For example, higher-point diagrams of any type are difficult to calculate because longitudinal polarizations and momenta introduce nonlinearities, and in particular cancellation of anomalies (or any ϵ-tensor contribution) is difficult to check.

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Covariant quantization of the Green-Schwarz action was attempted [1]. A class of derivative gauges was introduced that led to a pyramid of ghosts. Counting arguments showed that the conformal anomaly canceled, and summation of ghost determinants agreed with the lightcone result due to the "identity" $1 - 2 + 3 - \cdots = 1/4$. Unfortunately, due to a noninvertible transformation the gauge-fixed action found by this method proved not to be invariant under the Becchi-Rouet-Stora-Tyutin transformations derived by the same method [2]. This problem already appeared for the Casalbuoni-Brink-Schwarz superparticle.

In the meantime, an alternative approach to the quantum superparticle was developed [3], based on adding extra dimensions to the lightcone, a method that had successfully given free gauge-invariant actions for arbitrary representations of the Poincaré group in arbitrary dimensions [4]. This approach directly gave a BRST operator with the right cohomology. Using the relation between this BRST operator and Zinn-Justin-Batalin-Vilkovisky first-quantization [5], a manifestly supersymmetric classical mechanics action for this superparticle followed, including a BRST-invariant gauge-fixed action [6]. A crucial difference from the previous method was that "nonminimal" fields were required: There was necessarily a "pyramid" of ghosts, not just a linear tower. However, because of a required Fierz identity, the method of adding extra dimensions could not be directly applied to the lightcone Green-Schwarz superstring.

Various alternatives for a manifestly supersymmetric superstring have since been tried; the most successful is the pure spinor formalism [7]. It has proven somewhat more useful than RNS or lightcone GS approaches in calculating tree amplitudes [8]; its application to loop amplitudes is in progress [9]. If the formalism for all loops is developed, it should provide a simpler proof of finiteness, which previously required a combination of RNS and lightcone GS results (and equivalence of the two approaches). The pure spinor approach has two main shortcomings:

The first problem is the lack of a manifestly supersymmetric (and Lorentz covariant) path-integration measure. This is a problem in all known superspace approaches to first-quantizing superparticles and superstrings. One consequence is that Green functions (or the effective action in the superparticle case) are not manifestly supersymmetric off shell. Another is that gauge fixing the string field theory (with ghost fields) is not simple. We will not address this problem here.

The other problem is that the pure-spinor BRST operator lacks the c and b ghosts associated with the usual 2D coordinate invariances (and their associated Virasoro constraints). This is directly related to the lack of a corresponding action with worldsheet metric; the action is known only in the conformal gauge. Furthermore, the moduli that are the remnants of the metric in the conformal gauge must be inserted by hand. Another consequence of the lack of these ghosts as fundamental variables is that they must be reconstructed as complicated composite operators for use as insertions in loop diagrams. The (gauge-fixed) action, BRST operator, moduli, and operator insertions are thus separate postulates of the formalism, rather than all following from a gauge-invariant action as in other formalisms.

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In this paper we will formulate the superstring with the ghost structure indicated by the original attempt of [1] and the successful treatment of the superparticle in [3]: the usual c and b ghosts, and a pyramid of spinors labeled by ghost number and generation. The main result is the BRST operator (from which the gauge-invariant action follows), which takes the form

$$Q_{\text{sstring}} = U \left(\int c\, T + \frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi|_{>} \right) U^{-1} \tag{1.1}$$

with

$$U = e^{\int \tilde{\theta} D}\; e^{i\int R^a|_{>} P_a^{(\pm)}}\; e^{\int (R^{\oplus} + \theta\tilde{\gamma}^{\oplus}\theta/2)|_{>} b}\,, \tag{1.2}$$

where T is essentially the energy-momentum tensor, D and P are the usual "covariant derivatives" in the affine Lie algebra of the classical superstring, $\tilde{\theta}$ is a certain linear combination of ghost θ's, R^i are certain expressions quadratic in θ's, π is conjugate to θ, $\tilde{\gamma}^{\oplus}$ is a ghost partner to the gamma matrices γ^a (which act only on θ and π), and $|_{>}$ picks out the ghost contributions. The gauge-fixed hamiltonian is just $\{Q, \int b\} = \int T$. The unitary transformations are necessary because they change the Hilbert space, and so cannot be dropped: A simple analog is the BRST operator for the spinning (Dirac) particle in an external gauge field:

$$Q_{\text{Dirac}} = e^{c\gamma^a\nabla_a/\gamma^{\oplus}}(\gamma^{\oplus 2}b)e^{-c\gamma^a\nabla_a/\gamma^{\oplus}} = \gamma^{\oplus 2}b + \gamma^{\oplus}\gamma^a\nabla_a - \frac{1}{2}c(\gamma^a\nabla_a)^2\,,$$

where $1/\gamma^{\oplus}$ doesn't exist on the correct Hilbert space, but cancels when the "unitary" transformation is evaluated.

We begin in section 2 by reviewing the free superparticle, which has already been quantized (and its BRST cohomology checked) in this approach. Because of the similarity of the algebra of super Yang-Mills [10] to that of the superstring [11], in section 3 we couple this superparticle to external super Yang-Mills superfields. We use an almost identical method to derive the BRST operator for the superstring in section 4. We finish with our conclusions in section 5. (Mathematical details are relegated to the appendices A–E.)

2. Review of free superparticle

We will start from the free super BRST operator derived in [3]. The generic BRST operator for arbitrary fields (massless, or massive by dimensional reduction) is constructed by starting with a representation of the lightcone $SO(D-2)$ (which defines the theory) and adding 4 bosonic and 4 fermionic dimensions to obtain a covariant representation, including all auxiliary fields and ghosts. (This is somewhat redundant for bosons, but necessary for fermions.) The resulting generators S^{AB} of $OSp(D,2|4)$ spin carry vector indices A, B that are separated into the usual $SO(D-1,1)$ indices a, b and the rest as

$$A = (+,-,a;\mu,\tilde{\mu}) = (+,-,i)\,, \qquad \mu = (\oplus,\ominus)\,, \tag{2.1}$$

where $+,-$ belong to an $SO(1,1)$ subgroup and $\mu,\tilde{\mu}$ to two $Sp(2)$'s, of which only the diagonal subgroup will be useful. The BRST operator then takes the generic form

$$Q'_{\text{free}} = \frac{1}{2}c\,\Box + S^{\oplus a}\partial_a + S^{\oplus\oplus}b + S^{\tilde{\oplus}-} \qquad (\,\Box = \partial^a\partial_a\,)\,. \tag{2.2}$$

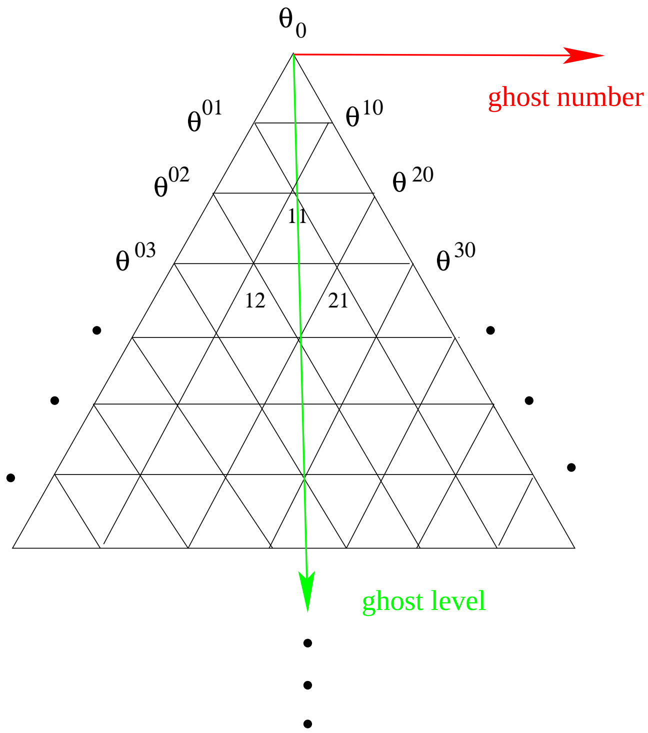

Figure 1: infinite pyramid of ghosts.

In the case of the superparticle, the spin operators are

$$S^{AB} = -\frac{1}{4}\,\bar{\eta}\Gamma^{[A}\Gamma^{B\}}\eta \tag{2.3}$$

in terms of self-conjugate variables η, which arose from the usual self-conjugate SO(D-2) fermionic spinor of lightcone superspace. We decompose the OSp$(D,2|4)$ gamma-matrices Γ^A in terms of those of the subgroup SO$(1,1)$ and those (γ) of the subgroup OSp$(D-1,1|4)$ as

$$\Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \qquad \Gamma^+ = \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^- = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix} \tag{2.4}$$

with (anti)commutation relations

$$\{\gamma^a,\gamma^b\} = -2\eta^{ab}\,, \qquad \eta^{ab} = (-+++\cdots)$$

$$\{\gamma^a,\gamma^\mu\} = \{\gamma^a,\tilde{\gamma}^\mu\} = 0 \tag{2.5}$$

$$[\gamma^\mu,\gamma^\nu] = [\tilde{\gamma}^\mu,\tilde{\gamma}^\nu] = 2C^{\mu\nu}\,, \qquad [\gamma^\mu,\tilde{\gamma}^\nu] = 0\,, \tag{2.6}$$

where $C^{\mu\nu}$ is the Sp(2) metric with convention

$$C^{\oplus\ominus} = C_{\ominus\oplus} = i = -C^{\ominus\oplus} = -C_{\oplus\ominus} \tag{2.7}$$

and we have denoted $\gamma^{\tilde{\mu}} \equiv \tilde{\gamma}^\mu$ for legibility. The generalization of the fermionic superspace coordinate θ and its conjugate momentum appear through the analogous decomposition

$$\eta = \begin{pmatrix} \pi \\ \theta \end{pmatrix}, \qquad \pi = \frac{\partial}{\partial\theta}\,. \tag{2.8}$$

We begin with a chiral (Weyl) spinor η, and multiplication by any Γ changes the chirality: not just Γ^a (γ^a) as usual, but also $\Gamma^\pm$, which shows that π and θ have opposite chirality (as expected, since they are conjugate), and Γ^μ (γ^μ).

This BRST operator is supersymmetric and also has an infinite pyramid of ghosts. To see these ghosts we need to define creation and annihilation operators from γ^μ and $\tilde{\gamma}^\mu$ as follows:

$$\begin{aligned} \gamma^\mu &= a^\mu + a^{\dagger\mu}\,, \qquad \tilde{\gamma}^\mu = i(a^\mu \; - \; a^{\dagger\mu}) &(2.9)\\ [a^\mu, a^{\dagger\nu}] &= C^{\mu\nu}\,. &(2.10)\end{aligned}$$

Then θ can be expanded giving the usual physical supersymmetry fermionic coordinate θ_0 at the top of the infinite pyramid of ghosts:

$$\begin{aligned} |^{p,q}\rangle &\equiv i^{\frac{(p+q)(p+q+1)}{2}} \frac{1}{\sqrt{p!}\sqrt{q!}} (a^{\dagger\oplus})^p (a^{\dagger\ominus})^q \;|0\rangle \\ \langle_{p,q}| &\equiv (-i)^{\frac{(p+q)(p+q+1)}{2}} \frac{1}{\sqrt{p!}\sqrt{q!}} \langle 0| \;(a_\oplus)^p (a_\ominus)^q \\ \langle_{p,q}|^{r,s}\rangle &= \delta_p^r \delta_q^s \\ \theta^{p,q} &\equiv \langle\theta|^{p,q}\rangle \;=\; \theta^{p,q\dagger} \\ \pi_{p,q} &\equiv \langle_{p,q}|\pi\rangle\,, \qquad (2.11)\end{aligned}$$

where

$$\theta_0 \;\equiv\; \theta^{0,0}\,. \qquad (2.12)$$

A power of i has been inserted to make $\theta^{p,q}$ real: The product of n real fermions gets a sign $(-1)^{n(n-1)/2}$ under hermitean conjugation, because of the reverse ordering. The ghost a's and $a^\dagger$'s are fermions, because they take fermions to bosons, and vice versa (in contrast to ordinary γ matrices, which take fermions to fermions). Thus $\theta^{p,q}$ is the product of $p+q+1$ fermions, including $\langle\theta|$ itself. Then $\pi_{p,q}$ is not necessarily hermitean, but has been defined to give 0 or 1 in graded commutators. (But $\pi^{p,q}$ is always hermitean, like $|\pi\rangle$ and π_0.) We will sometimes also use a notation where $\theta^{p,q}$ carries instead p $\oplus$'s and q $\ominus$'s: For example, $\theta^{1,0} \equiv \theta^\oplus$. Note that the ghosts alternate in both statistics and chirality with each ghost level.

So in superspace notation the free super BRST operator is

$$Q'_{\rm free} = \frac{1}{2}c\Box \; - \frac{1}{2}\bar{\pi}\gamma^{\oplus 2}\theta b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi \; - \frac{i}{2}\bar{\pi}\gamma^\oplus \not{p}\theta\,, \qquad \not{p} \;\equiv\; -i\partial_a\gamma^a\,. \qquad (2.13)$$

We can make a unitary transformation on $Q'_{\rm free}$ to give a convenient form with which to work. Specifically, the unitary transformation

$$Q_{\rm free} = U_0 Q'_{free} U_0^\dagger \qquad (2.14)$$

with

$$U_0 = e^{\bar{\theta}\tilde{\gamma}^\oplus\theta b/2} \qquad (2.15)$$

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gives Q_{free} in terms of the supersymmetry generator q_0, spinor covariant derivative d_0 and all their nonminimal versions:

$$Q_{\text{free}} = \frac{1}{2}c\Box - 2\bar{\pi}a^{\dagger\oplus}a^{\oplus}\theta b - \frac{i}{2}\bar{q}a^{\dagger\oplus}d \tag{2.16}$$

$$q = \pi - \not{p}\theta\,, \qquad d = \pi + \not{p}\theta\,. \tag{2.17}$$

Actually, q_0 is the only part of q that does not appear in this form of the BRST operator: Because of the creation and annihilation operators, θ_0 and π_0 appear only as their supersymmetry invariant combination d_0. Thus the supersymmetry generator that anti-commutes with this form of Q is just the usual one q_0. (This can also be derived in a straightforward way by starting with the lightcone q.) Then the supersymmetry generator for Q'_{free} can be obtained by inverting the unitary transformation on q_0:

$$\begin{aligned} q'_0 &= U_0^\dagger q_0 U_0 \\ &= \pi_0 - \not{p}\theta_0 - \theta^{\oplus}b\,. \end{aligned} \tag{2.18}$$

3. Interacting superparticle

The (D=3,4,6,10) superparticle BRST operator in a super Yang-Mills background (with constant superfield strength) is closely related to the superstring BRST operator. The introduction of the SYM background can be established by gauge covariantizing the super covariant derivatives p_a and $d_{0\alpha}$:

$$p_a \longrightarrow \nabla_a \tag{3.1}$$

$$d_0 \longrightarrow \nabla_0\,. \tag{3.2}$$

Then the graded algebra among the covariant derivatives is [10]

$$[\nabla_a, \nabla_b] = F_{ab} \tag{3.3}$$

$$\{\nabla_{0\alpha}, \nabla_{0\beta}\} = 2\gamma_{a\alpha\beta}\nabla^a \tag{3.4}$$

$$[\nabla_{0\alpha}, \nabla_a] = \gamma_{a\alpha\beta}W^{\beta}\,. \tag{3.5}$$

The Bianchi identity from the above algebra gives

$$\gamma_{a\alpha\beta}[\nabla^a, W^{\beta}] = 0 \tag{3.6}$$

and the D=3,4,6,10 dimensional gamma matrix (which is symmetric in those cases) identity

$$\gamma_{a(\alpha\beta}\gamma^a{}_{\gamma)}{}^{\delta} = 0\,. \tag{3.7}$$

We begin at linear order in the fields, where the background satisfies the equations of motion

$$\{\nabla_{\alpha}, W^{\alpha}\} = 0 \tag{3.8}$$

$$[\nabla^a, F_{ab}] = 0\,. \tag{3.9}$$

3.1 Constant YM background

One way to build this interacting super BRST operator is by considering an ordinary constant YM background first, and next supersymmetrizing it by including a constant fermionic field strength (not yet superfield) $\overset{\circ}{w}{}^{\alpha}$. Then we extend the result to a nonconstant SYM background in the next subsection.

Making the gauge choice

$$\overset{\circ}{A}_a = \frac{i}{2} x^b \overset{\circ}{F}_{ba}$$

for constant field strength, the super BRST operator can be written in the form

$$\overset{\circ}{Q}{}'_{\text{YMB}} = Q'_{\text{free}} + \tfrac{1}{2} \overset{\circ}{F}_{ab} V^{ab} \,. \tag{3.10}$$

We then find

$$V^{ab} \;=\; \frac{i}{2} c x^{[a} p^{b]} + \frac{i}{2} R^{\oplus} R^{[a} p^{b]} - \; (c + R^{\oplus}) \; \bar{\pi} \gamma^{ab} \theta + \frac{1}{4} (x + R)^{[a} \bar{\pi} \gamma^{\oplus} \gamma^{b]} \theta \tag{3.11}$$

in terms of an expression R^i defined below, where we use the notation

$$C^{[a} D^{b]} \;\equiv\; C^a D^b - C^b D^a \tag{3.12}$$

$$\gamma^{ab} \;\equiv\; -\frac{1}{4} \gamma^{[a} \gamma^{b]} \,. \tag{3.13}$$

We can also write

$$\begin{aligned} V^{ab} &= V^{\oplus ab} \\ V^{ijk} &= i(x^i x^j + R^i R^j) p^k + \tfrac{1}{2} (x + R)^{(i} \bar{\pi} \gamma^{j)} \gamma^k \theta \\ x^{\oplus} &= c \,, \qquad p^{\oplus} = 0 \,. \end{aligned} \tag{3.14}$$

(There is further antisymmetrization in the last two indices upon contraction with F, following the graded symmetrization in the first two indices shown above: The tensor V^{ijk} has mixed symmetry.)

The expression R^i is given by

$$R^i(\theta) \equiv \frac{1}{2} \bar{\theta} \mathcal{O} \gamma^i \theta \tag{3.15}$$

where the operator $\mathcal{O}$ is defined to satisfy

$$\begin{aligned} [\gamma^{\oplus}, \mathcal{O}] &= 0 \\ \{\tilde{\gamma}^{\oplus}, \mathcal{O}\} &= 2\gamma^{\oplus} \\ [a^{\dagger\oplus} a_{\oplus} - a^{\dagger\ominus} a_{\ominus}, \mathcal{O}] &= 0 \\ \langle 0 | \mathcal{O} \tilde{\gamma}^{\oplus} &= -i \langle 0 | \tilde{\gamma}^{\oplus} \;=\; \langle 0 | \gamma^{\oplus} \,. \end{aligned} \tag{3.16}$$

As an explicit form of $\mathcal{O}$ we find

$$\mathcal{O} = \frac{1}{2} \left\{ \frac{1}{\tilde{\gamma}^{\oplus}}, \gamma^{\oplus} \right\} \tag{3.17}$$

where

$$\frac{1}{\tilde{\gamma}^{\oplus}} = \sum_{p=0}^{\infty}\Big[\Theta_{N_\oplus - N_\ominus}\frac{N_\oplus !}{(N_\oplus + p+1)!}\; ia_\oplus (ia_\oplus a_\ominus)^p - \\ - a^{\dagger\ominus}(-ia^{\dagger\oplus}a^{\dagger\ominus})^p \Theta_{N_\ominus - N_\oplus}\frac{N_\ominus !}{(N_\ominus + p+1)!}\Big] \tag{3.18}$$

with

$$\Theta_x = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad , \qquad N_\mu = a^{\dagger\mu}a_\mu$$

(not summed over μ). This representation satisfies (3.16) if we regularize indefinite norm states. (See the appendices for details.)

3.2 Constant SYM background

From this $\overset{\circ}{Q}{}'_{YMB}$ we can construct a BRST operator for a supersymmetric constant SYM background $\overset{\circ}{Q}{}'_{SYMB}$ in the form

$$\overset{\circ}{Q}{}'_{\rm SYMB} = Q'_{\rm free} + \frac{1}{2}\overset{\circ}{F}_{ab}V^{ab} + \overset{\circ}{w}{}^\alpha V_\alpha \,. \tag{3.19}$$

In addition to first-quantized transformations we take q'_0 (2.18) to also generate the second-quantized transformations of $\overset{\circ}{w}{}^\alpha$ and $\overset{\circ}{F}_{ab}$

$$\{q'_{0\beta}, \overset{\circ}{w}{}^\alpha\} \;=\; \gamma^{ab}{}_\beta{}^\alpha \overset{\circ}{F}_{ab} \tag{3.20}$$

$$\{q'_0, \overset{\circ}{F}_{ab}\} \;=\; 0 \tag{3.21}$$

so that they cancel up to a gauge transformation (generated by Q'_{free}):

$$\{q'_0,\; \overset{\circ}{Q}{}'_{\rm SYMB}\} = \{Q'_{\rm free}, \Psi\}\,. \tag{3.22}$$

We then have

$$\frac{1}{2}\overset{\circ}{F}_{ab}\;\{q'_{0\beta}, V^{ab}\} \;-\; \overset{\circ}{w}{}^\alpha [q'_{0\beta}, V_\alpha] = -\overset{\circ}{F}_{ab}\;\gamma^{ab}{}_\beta{}^\alpha\; V_\alpha + \{Q'_{free}, \Psi_\beta\}\,. \tag{3.23}$$

This is true if we define V_α by

$$\{q'_{0\beta}, V^{ab}\}|_{\gamma^{ab}} \;\equiv\; -\,2\gamma^{ab}{}_\beta{}^\alpha\; V_\alpha \tag{3.24}$$

which means we define V_α from the left-hand side by selecting only terms with an explicit γ^{ab}.

With this definition we find V_α and Ψ_α

$$V_\alpha \;\equiv\; -\,(c + R^\oplus)\; q'_{0\alpha} \tag{3.25}$$

$$\Psi_\alpha \;\equiv\; -\frac{i}{2}\left(x^b + \; R^b\right)\left(\gamma^a_{\alpha\beta}\theta_0^\beta \overset{\circ}{F}_{ab} - 2\gamma_{b\alpha\beta}\overset{\circ}{w}{}^\beta\right) + \frac{1}{2}\left(\gamma^b_{\alpha\beta}\theta_0^\beta + \gamma^b_{\alpha\beta}\tilde{\theta}_0^\beta\right) iR^a \overset{\circ}{F}_{ab} \tag{3.26}$$

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where

$$\tilde{\theta} = -i\langle 0|\mathcal{O}|\theta\rangle = \langle 0|e^{ia_\oplus a_\ominus} - 1|\theta\rangle \tag{3.27}$$

which contains all nonminimal ghost-number-zero ghosts.

To obtain a gauge independent and explicitly supersymmetric expression we perform a unitary transformation

$$U_1 = e^\Lambda$$

where

$$\begin{aligned}\Lambda =\,& iR^b\left(\theta_0\gamma_b\overset{\circ}{w}+\tilde{\theta}\gamma_b\overset{\circ}{w}+\frac{1}{2}\theta_0\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac}+\frac{1}{2}\tilde{\theta}\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac}+\frac{1}{2}\tilde{\theta}\gamma_b\gamma^{ac}\tilde{\theta}\overset{\circ}{F}_{ac}-\frac{1}{2}\theta_0\gamma^c\tilde{\theta}\overset{\circ}{F}_{bc}\right)-\\ &-\theta_0\gamma^b\tilde{\theta}\left(\frac{1}{3}\theta_0\gamma_b\overset{\circ}{w}+\frac{2}{3}\tilde{\theta}\gamma_b\overset{\circ}{w}+\frac{1}{4}\theta_0\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac}+\frac{3}{4}\tilde{\theta}\gamma_b\gamma^{ac}\theta_0\overset{\circ}{F}_{ac}+\frac{5}{12}\tilde{\theta}\gamma_b\gamma^{ac}\tilde{\theta}\overset{\circ}{F}_{ac}\right).\end{aligned} \tag{3.28}$$

After another unitary transformation U_0 (2.15), $\overset{\circ}{Q}'_{SYMB}$ becomes (at the linearized level)

$$\begin{aligned}\overset{\circ}{Q}_{\mathrm{SYMB}} =\,& \frac{1}{2}\left(c+R^\oplus+\frac{1}{2}\bar{\theta}\tilde{\gamma}^\oplus\theta\right)|_>(\Box+W\nabla_0-\bar{\pi}\gamma^{ab}\theta|_>F_{ab})-\\ &-2\bar{\pi}a^{\dagger\oplus}a^\oplus\theta|_>b+\frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi|_>-\\ &-\frac{i}{2}\left(\nabla_a+\tilde{\theta}\gamma_aW+\frac{1}{2}\tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta}\right)\bar{\pi}\gamma^\oplus\gamma^a\theta|_>-\\ &-\frac{1}{2}\bar{\pi}^\oplus\nabla_0-\frac{1}{2}\left(\nabla_a+\tilde{\theta}\gamma_aW+\frac{1}{2}\tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta}-\frac{i}{2}R^b|_>F_{ab}\right)\bar{\theta}^\oplus\gamma^a\nabla_0-\\ &-\frac{1}{3}\bar{q}^\oplus\gamma^a\tilde{\theta}\tilde{\theta}\gamma_aW+\frac{5}{24}\bar{q}^\oplus\gamma^a\tilde{\theta}\tilde{\theta}\gamma_a\{W,\nabla_0\}\tilde{\theta}+\\ &+\frac{i}{4}\bar{q}^\oplus\gamma^b\tilde{\theta}R^a|_>F_{ab}+\\ &+\frac{1}{4}[i\nabla_bR^b|_>,-i\nabla_a\bar{\pi}\gamma^\oplus\gamma^a\theta|_>]+\\ &+R^\oplus|_>P_a\left(\tilde{\theta}\gamma^aW+\frac{1}{2}\tilde{\theta}\gamma^a\{W,\nabla_0\}\tilde{\theta}\right)+\\ &+\frac{1}{2\cdot 3!}\left[i\nabla_cR^c|_>,[i\nabla_bR^b|_>,-i\nabla_a\bar{\pi}\gamma^\oplus\gamma^a\theta|_>]\right]\end{aligned} \tag{3.29}$$

where $|_>$ means that we drop θ_0 contributions, and

$$\Box = -\nabla^a\nabla_a \tag{3.30}$$

$$\nabla_a = p_a + A_a \tag{3.31}$$

$$\nabla_{0\alpha} = \pi_{0\alpha} + (\not{p}\theta_0)_\alpha + A_\alpha \tag{3.32}$$

$$q^{\oplus\alpha} = \pi^{\oplus\alpha} + (\not{p}\theta^\oplus)^\alpha\,. \tag{3.33}$$

The superfields have the θ_0 expansions

$$F_{ab} = \overset{\circ}{F}_{ab} \tag{3.34}$$

$$W^\alpha = \overset{\circ}{w}^\alpha + (\gamma^{ab}\theta_0)^\alpha\overset{\circ}{F}_{ab} \tag{3.35}$$

$$
\begin{aligned}
A_a &= \overset{\circ}{A}_a + \bar{\theta}_0 \gamma_a \overset{\circ}{w} + \frac{1}{2} \bar{\theta}_0 \gamma_a \gamma^{bc} \theta_0 \overset{\circ}{F}_{bc} && (3.36)\\
A_\alpha &= (\gamma^a \theta_0)_\alpha \overset{\circ}{A}_a + \frac{2}{3} (\gamma^a \theta_0)_\alpha \bar{\theta}_0 \gamma_a \overset{\circ}{w} + \frac{1}{4} (\gamma^a \theta_0)_\alpha \bar{\theta}_0 \gamma_a \gamma^{bc} \theta_0 \overset{\circ}{F}_{bc} && (3.37)
\end{aligned}
$$

in the gauge

$$
\begin{aligned}
\overset{\circ}{A}_a &= \frac{i}{2} x^b \overset{\circ}{F}_{ba} && (3.38)\\
\overset{\circ}{A}_\alpha &= 0 && (3.39)
\end{aligned}
$$

used above, but (3.29) is manifestly gauge independent and supersymmetric.

3.3 Arbitrary SYM background

After making a final unitary transformation

$$
U_3 = e^{(R^\oplus + \theta \tilde{\gamma}^\oplus \theta / 2)|_> b}
$$

the above BRST operator can be written in the simple form

$$
\begin{aligned}
Q''_{\mathrm{SYMB}} &= U \left[\frac{1}{2} c \left(\Box + W \nabla_0 - \bar{\pi} \gamma^{ab} \theta|_> F_{ab} \right) + \frac{1}{4} \bar{\pi} \tilde{\gamma}^\oplus \pi|_> \right] U^{-1} \\
U &= e^{\tilde{\theta} \nabla_0} e^{i R^a|_> \nabla_a} e^{(R^\oplus + \theta \tilde{\gamma}^\oplus \theta/2)|_> b} && (3.40)
\end{aligned}
$$

which can be applied directly to the case of an arbitrary, nonlinear SYM background.

In fact, the nilpotence of this BRST operator does not seem to require that the background be on shell. This is contradictory to the usual result that any description of linearized "quantum" Yang-Mills in a Yang-Mills background must have the background on shell, since nonabelian gauge invariance relates kinetic and interaction terms [12]. (Similar remarks apply to any nonabelian gauge theory, such as gravity or strings.) This paradox is probably due to the fact that we have not required an "integrability" condition on the background: For a generic self-interacting field theory, an action (or ZJBV action) of the form

$$
S = \tfrac{1}{2} \phi^j \phi^i K_{ij} + \tfrac{1}{6} \phi^k \phi^j \phi^i V_{ijk} + \cdots \qquad (3.41)
$$

results in the kinetic operator (or BRST operator) in a background

$$
Q_{ij} = K_{ij} + \phi^k V_{kij} + \cdots . \qquad (3.42)
$$

From S we can see that $K, V, \ldots$ must be totally (graded) symmetric. In Q, this condition on K is seen to follow simply from hermiticity, but the condition on V is not so obvious. Since we are ultimately concerned with the BRST operator for the superstring without background, and are using the SYM case in a background only as an analogy, we will not consider this obscurity further here.

As explained in the Introduction, in the above expression for the BRST operator (3.40) we are not allowed to remove the exponential factors, since that would lead to a trivial result. This fact can be understood already in the free case: The BRST operator that

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would result from dropping the background and exponentials has the wrong cohomology, since the remaining two terms have no dependence on θ_0, so one would obtain an ordinary superfield satisfying only the Klein-Gordon equation. In this case the exponentials are required for Q to be regularizable: Certain poorly defined quantities cancel upon their expansion. (See appendices B and C.)

4. Superstring

The superstring is described by a 2D field theory whose algebra of covariant derivatives (currents) resembles that of interacting particle covariant derivatives for a constant SYM background:

$$\begin{aligned}
\left\{D^{(\pm)}_\alpha(1), D^{(\pm)}_\beta(2)\right\} &= 2\delta(2-1)\gamma^a_{\alpha\beta}P^{(\pm)}_a(1) \\
\left[D^{(\pm)}_\alpha(1), P^{(\pm)}_a(2)\right] &= 2\delta(2-1)\gamma_{a\alpha\beta}\Omega^{(\pm)\beta}(1) \\
\left\{D^{(\pm)}_\alpha(1), \Omega^{(\pm)\beta}(2)\right\} &= \pm i\delta'(2-1)\delta^\beta_\alpha \\
\left[P^{(\pm)}_a(1), P^{(\pm)}_b(2)\right] &= \pm i\delta'(2-1)\eta_{ab} \\
\left[P^{(\pm)}, \Omega^{(\pm)}\right] &= \{\Omega^{(\pm)}, \Omega^{(\pm)}\} \;=\; 0\,,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
D^{(\pm)}_\alpha &= \pi_{0\alpha} + (\gamma^a\theta_0)_\alpha \hat{P}^{(\pm)}_a \pm i\frac{1}{2}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\theta'_0 \\
P^{(\pm)}_a &= \hat{P}^{(\pm)}_a \pm i\theta_0\gamma_a\theta'_0 \\
\Omega^{(\pm)\alpha} &= \pm i\theta'_0
\end{aligned} \tag{4.2}$$

and

$$\hat{P}^{(\pm)} = \frac{1}{\sqrt{2}}\left(i\frac{\delta}{\delta X} \pm X'\right) \tag{4.3}$$

in the hamltonian formalism correspond to the left(right)-moving combinations of P_0 and P_1 of the first-order formalism after using the equation of motion for P_1 (see below). (In the definitions above, $(\pm)$'s on π and θ are understood.) Also, $'$ means a σ derivative as usual. D, P, Ω (anti)commute with the supersymmetry generator

$$q_\alpha \;=\; q^{(+)}_\alpha + q^{(-)}_\alpha\,, \qquad q^{(\pm)}_\alpha \;=\; \int \pi_{0\alpha} - (\gamma^a\theta_0)_\alpha\hat{P}^{(\pm)}_a \mp i\frac{1}{6}(\gamma^a\theta_0)_\alpha\theta_0\gamma_a\theta'_0\,. \tag{4.4}$$

So we can see the analogy between the covariant derivatives of the free superstring and the superparticle with SYM background.

$$(D_\alpha, P_a, \Omega^\alpha) \leftrightarrow (\nabla_\alpha, \nabla_a, W^\alpha) \tag{4.5}$$

as well as the less precise analogy

$$' \leftrightarrow F_{ab}\,. \tag{4.6}$$

4.1 BRST

Now we can guess the result for the superstring BRST operator from the result of the superparticle in a constant SYM background:

$$Q_{\text{sstring}} = U\left(\int c\, T + \frac{1}{4}\bar\pi\tilde\gamma^{\oplus}\pi|_{>} \right) U^{-1}\,, \tag{4.7}$$

where

$$U = e^{\int \tilde\theta D}\ e^{\int iR^a|_{>} P_a}\ e^{\int (R^{\oplus}+\theta\tilde\gamma^{\oplus}\theta/2)|_{>} b} \tag{4.8}$$

and $Q_{\text{sstring}} = Q^{(+)}_{\text{sstring}} + Q^{(-)}_{\text{sstring}}$. From now on we will suppress the σ-integral symbol for convenience. Since θ_0 and π_0 appear only in D, P, and T, this Q is automatically supersymmetric under the above supersymmetry generator.

There are two major differences in T as compared to the superparticle: Firstly the string has a $c'b$ ghost contribution. Secondly the superstring has ΩD as an analog of $W\nabla_0 - \bar\pi\gamma^{ab}\theta F_{ab}$, from the correspondence above. So our trial form of T is

$$\begin{aligned} T^{(\pm)} = \frac{1}{2}\,\Box_{\pm} \mp i\Big\{ & c'b + \bar\theta'\pi + w^{\pm}(\bar\theta\pi)' + \\ & + A_1^{\pm}\left[\bar\theta(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi\right]' + A_2^{\pm}\left[\bar\theta(a^{\dagger\oplus}a_{\oplus} + a^{\dagger\ominus}a_{\ominus})\pi\right]' \Big\} \end{aligned} \tag{4.9}$$

where $\Box_{\pm} = -\hat P^{(\pm)a}\hat P^{(\pm)}_a$. (The true energy-momentum tensor is actually $T \mp i(cb)'$.) The constants $w^{\pm}$, $A_1^{\pm}$ and $A_2^{\pm}$ will be determined by 3 conditions: (1) The conformal weight of $\bar\pi\tilde\gamma^{\oplus}\pi$ should be 1. (2) The conformal anomaly should cancel in $D = 10$. (3) θ_0 should have conformal weight 0 due to supersymmetry. (The A_1 term is ghost number, while the A_2 term is ghost level.)

Satisfying these constraints we find

$$\begin{aligned} A_1^{\pm} &= 1\,, \qquad A_2^{\pm} \;=\; w^{\pm} \;=\; 0 \\ \Rightarrow\quad T^{(\pm)} &= \frac{1}{2}\,\Box_{\pm} \mp i\left\{ c'b + \bar\theta'\pi + \left[\bar\theta(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi\right]' \right\} \end{aligned} \tag{4.10}$$

and the gauge-fixed hamiltonian is $\{Q, \int b^{(+)} + b^{(-)}\} = \int T^{(+)} + T^{(-)}$.

This Q has four interesting quantum numbers:(1) ghost number; (2) conformal weight, which is "momentum number" (1 for $P, b, \pi, '$) minus ghost number; (3) (10D) engineering dimension (-1 for x, c, $-\frac{1}{2}$ for θ, 2 for $'$); and (4) a mysterious "field weight", which is 1 for all fields, but for which we attribute a 1 for $\tilde\gamma^{\oplus}$. (Thus, Q is quadratic in momenta and primes, and cubic in fields and $\tilde\gamma^{\oplus}$'s.)

From now on let's concentrate on one chirality. After expanding the exponential factor and regularizing (as explained in appendix C) we find

$$\begin{aligned} Q^{(+)}_{\text{sstring}} = & \left(c + R^{\oplus} + \frac{1}{2}\bar\theta\tilde\gamma^{\oplus}\theta \right)|_{>} \times \\ & \times \left(\frac{1}{2}\Box_{+} \ - ic'b \ - \ i\bar\theta'\pi \ - \ i[\bar\theta(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]' \right) - \end{aligned}$$

$$
\begin{aligned}
&- 2\ \bar\pi a^{\dagger\oplus} a^{\oplus}\theta|_{>} b + \frac{1}{4}\ \bar\pi\tilde\gamma^{\oplus}\pi|_{>} - \\
&- \frac{i}{2}\ \left(\hat P_a\ + i\theta_0\gamma_a\theta_0'\ + 2i\tilde\theta\gamma_a\theta_0' + i\tilde\theta\gamma_a\tilde\theta'\ - \frac{1}{2}\ R_a'|_{>}\right)\ \bar\pi\gamma^{\oplus}\gamma^a\theta|_{>} - \\
&- \frac{1}{2}\ \bar\pi^{\oplus}\ \left(\pi_0 + (\gamma^a\theta_0)\hat P_a + i\frac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta_0'\ \right) - \\
&- \frac{1}{2}\ \left(\hat P_a\ + i\theta_0\gamma_a\theta_0'\ + 2i\tilde\theta\gamma_a\theta_0' + i\tilde\theta\gamma_a\tilde\theta'\ - \frac{1}{2}\ R_a'|_{>}\right) \times \\
&\times \bar\theta^{\oplus}\gamma^a \left(\pi_0 + (\gamma^b\theta_0)\hat P_b + i\frac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta_0'\ \right) - \\
&- \frac{1}{3}\ \bar q^{\oplus}\gamma^a\tilde\theta \left(2i\tilde\theta\gamma_a\theta_0' + i\frac{5}{4}\ \tilde\theta\gamma_a\tilde\theta' - \frac{3}{4}\ R_a'|_{>}\right) + \\
&+ \frac{1}{2}\ \left(\hat P_a\ + i\theta_0\gamma_a\theta_0'\ + 2i\tilde\theta\gamma_a\theta_0' + i\tilde\theta\gamma_a\tilde\theta'\ -\ \frac{1}{2}R_a'|_{>}\right)^2 R^{\oplus}|_{>} + \\
&+ ic'b\ \left(R^{\oplus} + \frac{1}{2}\bar\theta\tilde\gamma^{\oplus}\theta\right)|_{>}\,. \qquad (4.11)
\end{aligned}
$$

This Q satisfies $Q^2 = 0$, as can be checked directly.

4.2 Constraints

The constraints of the gauge-invariant action (see following subsection) can be obtained directly from the BRST operator by taking its (graded) commutator and keeping just ghost-number-zero terms: The Virasoro constraints $\mathcal{A}$ follow as usual from b (with the gauge-fixed action from $\int b$), while generalizations $\mathcal{B}$ of the $\gamma \cdot pd$ constraint (κ symmetry generator) follow from $\theta^{p,p+1}$, and first-class generalizations $\mathcal{E}$ of the second-class constraint d follow from $\pi^{p,p+1}$ [6]:

$$
\mathcal{A} = \frac{1}{2}\Box_{+} - i\sum_{q=0}^{\infty}\bar\theta'^{q,q}\pi_{q,q} \qquad (4.12)
$$

$$
\mathcal{B}_0 = \gamma^a\Pi\mathcal{P}_a(\boldsymbol{d}_0 + \pi^{1,1}) - 2\theta^{1,1}\mathcal{A} + 2\vartheta^0\left(\frac{1}{2}\mathcal{P}^2 + \mathcal{A}\right) \qquad (4.13)
$$

$$
\mathcal{B}_p = \gamma^a\Pi\mathcal{P}_a(\pi^{p,p} + \pi^{p+1,p+1}) +\ 2(\theta^{p,p} - \theta^{p+1,p+1})\mathcal{A} + 2\vartheta^p\left(\frac{1}{2}\mathcal{P}^2 + \mathcal{A}\right) \qquad (4.14)
$$

$$
\mathcal{E}_0 = \boldsymbol{d}_0 - \pi^{1,1} + \gamma^a\Pi\mathcal{P}_a\theta^{1,1} \qquad (4.15)
$$

$$
\mathcal{E}_p = \Pi(\pi^{p,p} - \pi^{p+1,p+1}) + \gamma^a\Pi\mathcal{P}_a(\theta^{p,p} + \theta^{p+1,p+1}) \qquad (4.16)
$$

where

$$
\begin{aligned}
\mathcal{P}_a &\equiv \hat P_a\ + i\theta_0\gamma_a\theta_0'\ + 2i\tilde\theta\gamma_a\theta_0' + i\tilde\theta\gamma_a\tilde\theta'\ - \frac{1}{2}\ \tilde R_a'|_{>} \\
\boldsymbol{d}_0 &\equiv \pi_0 + \gamma^a\hat P_a\theta_0 + i\frac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta_0'\ + \frac{2}{3}\gamma^a\tilde\theta\left(2i\tilde\theta\gamma_a\theta_0' + i\frac{5}{4}\ \tilde\theta\gamma_a\tilde\theta' - \frac{3}{4}\ \tilde R_a'|_{>}\right) \\
\vartheta^p &\equiv 2\theta^{1,1} + 2\theta^{2,2} + \cdots + 2\theta^{p-1,p-1} + \theta^{p,p} - \theta^{p+1,p+1} - 2\theta^{p+2,p+2} - 2\theta^{p+3,p+3} - \cdots \qquad (4.17)
\end{aligned}
$$

and $\tilde R^a$ indicates that only ghost-number-zero θ's are selected.

Since this procedure requires the component expression, we explicitly use the projection operator

$$\Pi = \frac{1}{\tilde{\gamma}^{\oplus}}(\tilde{\gamma}^{\oplus})_{\rm reg} \tag{4.18}$$

as, e.g., $\Pi|\theta\rangle$, in some terms, as explained in appendix C. Also, because $R^{\oplus}$ only interacts with $\pi\Pi$ we express $[\pi^{p,p+1}, R^{\oplus}]$ as a projected expression ϑ^p. (For the full expression, see appendix D.)

These constraints are closed classically (after regularization: see appendix D)

$$\begin{aligned}
[\mathcal{A}(1), \mathcal{A}(2)] &= -\delta'(2-1)[\mathcal{A}(1) + \mathcal{A}(2)] \\
[\mathcal{A}(1), \mathcal{E}_p(2)] &= -\delta'(2-1)\mathcal{E}_p(1) \\
[\mathcal{A}(1), \mathcal{B}_p(2)] &= -\delta'(2-1)[\mathcal{B}_p(1) + \mathcal{B}_p(2)] \\
\{\mathcal{E}_p(1), \mathcal{E}_q(2)\} &= 0 \\
\{\mathcal{B}_p^\alpha(1), \mathcal{B}_q^\beta(2)\} &= 8\delta'(2-1)\vartheta^{p\alpha}\mathcal{P}_a(1)\gamma^{a\beta\delta}E_{q\delta}(2) + \\
&\quad + 4\delta'(2-1)\gamma_a^{\alpha\lambda}(\pi^{p,p} + \pi^{p+1,p+1})_\lambda(1)\gamma^{a\beta\delta}E_{q\delta}(2) - \\
&\quad - 2\delta'(2-1)(\theta^{p,p} - \theta^{p+1,p+1} + \vartheta^p)^\alpha\mathcal{B}_q^\beta[(1) + (2)] + \\
&\quad + ((p,\alpha,1) \leftrightarrow (q,\beta,2)) \\
\{\mathcal{E}_{p\alpha}(1), \mathcal{B}_q^\delta(2)\} &= -4\delta_\alpha^\beta\delta(2-1)\mathcal{A}(1) - \\
&\quad - 2\delta'(2-1)(\theta^{q,q} - \theta^{q+1,q+1} + \vartheta^p)^\beta\mathcal{E}_{p\alpha}(2)\,,
\end{aligned} \tag{4.19}$$

where

$$E_q \equiv \sum_r i^{r-q}\frac{1}{\sqrt{r}}(\Theta_{r-q-1} - \Theta_{r-q-2})\mathcal{E}_{r-1}\,.$$

4.3 Action

From this hamiltonian form of the BRST operator for the superstring we can find the ZJBV form [5]:

$$Q^{\rm ZJBV} = \frac{i}{2}\dot{\phi}^A\phi_A - H - \check{\phi}_A\{\phi^A, Q^H]\,, \tag{4.20}$$

where

$$\begin{aligned}
[\phi^A, \phi^B\} &= \Omega^{AB} \\
\phi^A &= \Omega^{AB}\phi_B\,, \qquad \phi_B = \phi^A\Omega_{AB}
\end{aligned} \tag{4.21}$$

and $\check{\phi}_A$ is the antifield which is canonically conjugate to the field ϕ^A by the antibracket. ZJBV is useful for lagrangian quantization, but since Q is sufficient for hamiltonian quantization, we leave the details for appendix E.

Constraints appearing in the gauge-invariant hamiltonian have ghost number 0; their ghosts have ghost number 1; their antighosts have ghost number -1; the antifields of their antighosts have ghost number 0, and we can identify them in the ZJBV BRST operator with the Lagrange multipliers of the gauge-invariant hamiltonian. (More generally, we interpret all the negative-ghost-number fields as antifields.) Let

$$\Phi_{p,p+1} \equiv (-1)^{p+1}i\frac{1}{2}\sqrt{p+1}\check{\theta}_{p,p+1}$$

$$\begin{aligned}\Psi_{p,p+1} &\equiv (-1)^{p+1} i\frac{1}{2}\sqrt{p+1}\check{\pi}_{p,p+1} \\ \check{b} &\equiv g\end{aligned} \tag{4.22}$$

and similarly for their antifields. Then we find the gauge invariant action in hamiltonian form S_H either from the usual hamiltonian procedure (using the constraints of the previous subsection), or as the antifield-free part of $Q^{ZJBV}_{sstring}$:

$$S_H = -\dot{X}\hat{P}^0 + i\sum_{\pm,p}\dot{\theta}^{p,p}\pi_{p,p} - \sum_{\pm} g_\pm \mathcal{A}_\pm + \sum_{\pm,p\geq 0}\left(\tilde{\Phi}_{\pm,p,p+1}\mathcal{E}_p + \tilde{\Psi}_{\pm,p,p+1}\mathcal{B}_p\right) \tag{4.23}$$

(Again, for each sign $\pm$ we use fermions of the corresponding chirality.)

If we consider only the quadratic terms and the $\mathcal{A}$ term, keeping only the physical fields, and introducing $\hat{P}^1$ as an independent variable, we find the first-order, $2D$ world-sheet covariant form [11] (with world-sheet metric $\eta_{mn} = (-+)$)

$$S_0^{\rm phys} = \hat{P}^m\partial_m X - \frac{1}{2}g_{mn}\hat{P}^m\hat{P}^n + i\sqrt{2}\sum_{\pm}\partial_\pm\theta_0^\pm\pi_0^\pm \tag{4.24}$$

where $\theta_0^\pm \equiv \theta_{0L,R}$, $\pi_0^\pm \equiv \pi_{0L,R}$, and $\partial_\pm \equiv \frac{1}{\sqrt{2}}({e_0}^m \pm {e_1}^m)\partial_m$. By introducing supersymmetric variables

$$\begin{aligned}P^m &= \hat{P}^m + \epsilon^{mn}\left(\eta^+_{(0)n} - \eta^-_{(0)n}\right) \\ D_0^\pm &= \pi_0^\pm + \left[\hat{P}^\pm \pm \frac{1}{2}(\eta^+_{(0)\mp} - \eta^-_{(0)\mp})\right]\cdot\gamma\theta_0^\pm \\ \eta^\pm_{(0)m} &\equiv \frac{i}{\sqrt{2}}(\partial_m\theta_0^\pm)\gamma\theta_0^\pm\end{aligned} \tag{4.25}$$

(where we suppress spacetime indices for simplicity) and plugging this into (4.24) we find

$$\begin{aligned}S_0^{\rm phys} = &-\frac{1}{2}g_{mn}P^mP^n + P^m\left[\partial_m X - (\eta^+_{(0)m} + \eta^-_{(0)m})\right] - \\ &-\epsilon^{mn}\left[(\partial_m X)\cdot(\eta^+_{(0)n} + \eta^-_{(0)n}) - \eta^+_{(0)m}\eta^-_{(0)n}\right] + i\sqrt{2}\sum_{\pm}\partial_\pm\theta_0^\pm D_0^\pm\,.\end{aligned} \tag{4.26}$$

Except for the last term this is the Green-Schwarz supersting action. To extend this redefinition to the whole action one can further define (we use p, q for ghost level and $(p), (q)$ when there is confusion with world sheet indices l, m, n)

$$\begin{aligned}\mathcal{P}^m &= \hat{P}^m + \epsilon^{mn}(\eta^+_{(0)n} - \eta^-_{(0)n}) + \epsilon^{mn}(\chi_n^+ - \chi_n^-) \\ \mathcal{D}_0^\pm &= \pi_0^\pm + \left\{\left[\hat{P}^\pm \pm \frac{1}{2}(\eta^+_{(0)\mp} - \eta^-_{(0)\mp})\right]\cdot\gamma\theta_0^\pm \mp \frac{2}{3}(\xi^+_\mp - \xi^-_\mp)\cdot\gamma\tilde{\theta}^\pm\right\} \\ \mathcal{D}_p^\pm &= \pi_p^\pm + \mathcal{P}^\pm\cdot\gamma\theta_p^\pm \qquad (p\geq 1) \\ \pi_p^\pm &\equiv \pi^{p,p}_{L,R} \\ \theta_p^\pm &\equiv \theta^{p,p}_{L,R} \\ \vartheta_p^\pm &\equiv \vartheta^p_{L,R}\end{aligned}$$

$$
\begin{aligned}
\chi^{\pm}_m &\equiv \frac{1}{\sqrt{2}}\left(2i(\partial_m\theta^{\pm}_0)\gamma\tilde{\theta}^{\pm} - i(\partial_m\tilde{\theta}^{\pm})\gamma\tilde{\theta}^{\pm} + \frac{1}{2}\partial_m\tilde{R}^{\pm}\right)\\
\xi^{\pm}_m &\equiv \frac{1}{\sqrt{2}}\left(2i(\partial_m\theta^{\pm}_0)\gamma\tilde{\theta}^{\pm} - i\frac{5}{4}(\partial_m\tilde{\theta}^{\pm})\gamma\tilde{\theta}^{\pm} + \frac{3}{4}\partial_m\tilde{R}^{\pm}\right)\\
\eta^{\pm}_{(p)m} &\equiv \frac{i}{\sqrt{2}}(\partial_m\theta^{\pm}_p)\gamma\theta^{\pm}_p\\
\Phi_{p\pm} &\equiv \Phi^{L,R}_{p,p+1}\\
\Psi_{p\pm} &\equiv \Psi^{L,R}_{p,p+1}
\end{aligned} \tag{4.27}
$$

Then our manifestly worldsheet-covariant action reads

$$
S_0 = \tilde{S}_{\mathrm{GS}} + i\sqrt{2}\sum_{\pm,p\geq 0}\partial_{\pm}\theta^{\pm}_p\mathcal{D}^{\pm}_p + S_A\,, \tag{4.28}
$$

where S_A consists of Lagrange multipliers times all the (first-class) constraints other than Virasoro

$$
S_A \;=\; \sum_{\pm,p\geq 0}\left(\Psi_{p\pm}\mathcal{B}^{\pm}_p + \Phi_{p\pm}\mathcal{E}^{\pm}_p\right) \tag{4.29}
$$

and $\tilde{S}_{GS}$ is an extension of the usual GS action to the fields $\theta^{\pm}_p$ at nonzero ghost levels

$$
\begin{aligned}
\tilde{S}_{GS} = &-\frac{1}{2}g_{mn}\mathcal{P}^m\mathcal{P}^n +\\
&+\sum_{\pm}\mathcal{P}^{\pm}\left[\partial_{\pm}X - (\eta^{+}_{(0)\pm} + \eta^{-}_{(0)\pm}) + \sum_{p\geq 1}\eta^{\pm}_{(p)\pm} \pm (\chi^{+}_{\pm} - \chi^{-}_{\pm})\right] -\\
&-\epsilon^{mn}\left[(\partial_m X)\cdot\left\{(\eta^{+}_{(0)n} - \eta^{-}_{(0)n}) + (\chi^{+}_n - \chi^{+}_n)\right\} + \eta^{+}_{(0)m}\eta^{-}_{(0)n}\right] +\\
&+\frac{2}{3}\sum_{\pm}\pm\frac{i}{\sqrt{2}}(\partial_{\pm}\theta^{\pm}_0\gamma\tilde{\theta}^{\pm})(\xi^{+}_{\mp} - \xi^{-}_{\mp})\,.
\end{aligned} \tag{4.30}
$$

This is a first-order action in terms of the coordinates $X, \theta^{\pm}_p$, momenta $\mathcal{P}^m, \mathcal{D}^{\pm}_p$, worldsheet metric g_{mn}, and Lagrange multipliers $\Phi_{p\pm}, \Psi_{p,\pm}$. Now $\mathcal{E}_p$ and $\mathcal{B}_p$ are expressed in terms of these new variables as

$$
\begin{aligned}
\mathcal{E}^{\pm}_p &= \mathcal{D}^{\pm}_p - \mathcal{D}^{\pm}_{p+1} + 2\mathcal{P}^{\pm}\cdot\gamma\theta^{\pm}_{p+1}\\
\mathcal{B}^{\pm}_p &= \mathcal{P}^{\pm}\cdot\gamma(\mathcal{D}^{\pm}_p + \mathcal{D}^{\pm}_{p+1} - 2\mathcal{P}^{\pm}\cdot\gamma\theta^{\pm}_{p+1}) +\\
&\quad + (\Theta_{p-1}\theta^{\pm}_p - \theta^{\pm}_{p+1} + \vartheta^{\pm}_p)\times\\
&\quad \times\Bigg\{\mathcal{P}^{\pm 2} - [\mathcal{P}^{\pm} \mp (\eta^{+}_{(0)\mp} - \eta^{-}_{(0)\mp} + \chi^{+}_{\mp} - \chi^{-}_{\mp})]^2 \pm\\
&\qquad \pm\frac{1}{\sqrt{2}}\Bigg[\sum_{q\geq 1} i\partial_{\pm}\theta^{\pm}_q\left(\mathcal{D}^{\pm}_q - \mathcal{P}^{\pm}\cdot\gamma\theta^{\pm}_q\right) + i\partial_{\pm}\theta^{\pm}_0\times\\
&\qquad\quad \times\left(\mathcal{D}^{\pm}_0 - \left\{[\hat{P}^{\pm} \pm \frac{1}{2}(\eta^{+}_{(0)\mp} - \eta^{-}_{(0)\mp})]\cdot\gamma\theta^{\pm}_0 \mp \frac{2}{3}(\xi^{+}_{\mp} - \xi^{-}_{\mp})\cdot\gamma\tilde{\theta}^{\pm}\right\}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{4.31}
$$

Elimination of $\mathcal{P}^1$ by its equation of motion reproduces the previous hamiltonian form of the action except for terms quadratic in $\mathcal{E}$, which can be eliminated by a redefinition of Φ. The gauge-fixed action with ghosts is most easily obtained from the hamiltonian formalism as $H = \{Q, \int b\} = \int T$: Then (with the full $\theta^{p,q}$)

$$S_{\rm GF} \;=\; \hat{P}^m \partial_m X - \frac{1}{2}\eta_{mn}\hat{P}^m\hat{P}^n + i\sqrt{2}\sum_{\pm} \partial_\pm c^\pm b_{\pm\pm} + i\sqrt{2}\sum_{\pm}\partial_\pm \theta^\pm \pi^\pm \,. \tag{4.32}$$

5. Future

We have given a gauge-invariant action for the superstring and its corresponding BRST operator. The BRST-invariant gauge-fixed action is the obvious quadratic expression following from $\{Q, \int b\}$ (and is thus BRST invariant since $Q^2 = 0$). This is sufficient to perform S-matrix calculations (with vertex operators of the type given for the superparticle above), but a naive application would require a measure that breaks manifest supersymmetry. (For example, solving for the cohomology of the superparticle with this BRST operator in [3] required using the equivalent of the lightcone gauge.) In principle, a covariant measure that avoids picture changing altogether (in particular, for the bosonic ghosts) can be found by methods similar to those used in [4]; we hope to return to this problem. The cohomology of this BRST operator should also be checked: The massless level follows from the previous analysis for the superparticle; the massive levels should follow from a similar lightcone analysis.

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A. Sp(2) components

The matrix elements of the Sp(2) operators are

$$\langle p,q|\gamma^\oplus|r,s\rangle \;=\; C^{r,s}_{p,q}\left(\sqrt{p}\;\delta_{p,r+1}\delta_{q,s} + i\sqrt{q+1}\;\delta_{p,r}\delta_{q+1,s}\right) \tag{A.1}$$

$$\langle p,q|\tilde\gamma^\oplus|r,s\rangle \;=\; C^{r,s}_{p,q}\left(-i\sqrt{p}\;\delta_{p,r+1}\delta_{q,s} \;-\; \sqrt{q+1}\;\delta_{p,r}\delta_{q+1,s}\right), \tag{A.2}$$

where $C^{r,s}_{p,q} = i^{\frac{(r+s)(r+s+1)-(p+q)(p+q+1)}{2}}$. From these we can find "inverse" operators, especially

$$\langle p,q|\frac{1}{\tilde\gamma^\oplus}|r,s\rangle \;=\; i^{r-p}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,1}\left[\Theta_{p-q}\Theta_{s-q} \;-\; \Theta_{s-r}\Theta_{p-r}\right]. \tag{A.3}$$

It satisfies

$$\tilde\gamma^\oplus\frac{1}{\tilde\gamma^\oplus}\tilde\gamma^\oplus = \tilde\gamma^\oplus\,, \qquad \tilde\gamma^\oplus\frac{1}{\tilde\gamma^\oplus}\to I\,, \qquad \frac{1}{\tilde\gamma^\oplus}\tilde\gamma^\oplus \to I\,. \tag{A.4}$$

The arrows means there is cancellation among the multiplied matrix elements to the infinite ghost level. The subtle point of this cancellation will be studied in appendix B.

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Then we find that $\gamma^{\oplus}$ and $\frac{1}{\tilde{\gamma}^{\oplus}}$ don't (anti)commute but give

$$
\begin{aligned}
\langle p,q|\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}|r,s\rangle &= i^{r-p+1}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,0}[(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{s-q}+\Theta_{s-q-1})-\\
&\qquad -(\Theta_{s-r-1}+\Theta_{s-r})(\Theta_{p-r-1}+\Theta_{p-r})]\\
\langle p,q|\left[\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right]|r,s\rangle &= i^{r-p+1}C^{r,s}_{p,q}\sqrt{\frac{p!s!}{q!r!}}\;\delta_{q-p+r-s,0}[(\Theta_{p-q}-\Theta_{p-q-1})(\Theta_{s-q}+\Theta_{s-q-1})-\\
&\qquad -(\Theta_{s-r-1}-\Theta_{s-r})(\Theta_{p-r-1}+\Theta_{p-r})]\\
&= 2i\delta_{p,q}\delta_{r,s}\,. \qquad (A.5)
\end{aligned}
$$

Some interesting and useful commutators are

$$
\begin{aligned}
\langle p,q|\left[\gamma^{\oplus},\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}\right]|r,s\rangle &= -4i^{r-p}C^{r,s}_{p,q}\left(\sqrt{r+1}\delta_{p,q}\delta_{r+1,s}+\sqrt{p}\delta_{p,q+1}\delta_{r,s}\right) \qquad (A.6)\\
\langle p,q|\left[\gamma^{\oplus}\gamma^{\oplus},\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}\right]|r,s\rangle &= -8i^{r-p+1}C^{r,s}_{p,q}\Big(\sqrt{(r+2)(r+1)}\delta_{p,q}\delta_{r+2,s}+\\
&\qquad +\sqrt{p(p-1)}\delta_{p,q+2}\delta_{r,s}-\\
&\qquad -2\sqrt{p(r+1)}\delta_{p,q+1}\delta_{r+1,s}\Big)\\
\langle p,q|\left[\gamma^{\oplus}\gamma^{\oplus},\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}\right]\gamma^{\oplus}|r,s\rangle &= 16i^{r-p}C^{r,s}_{p,q}\Big(\sqrt{(r+3)(r+2)(r+1)}\delta_{p,q}\delta_{r+3,s}+\\
&\qquad +\sqrt{p(p-1)(r+1)}\delta_{p,q+2}\delta_{r+1,s}-\\
&\qquad -2\sqrt{p(r+2)(r+1)}\delta_{p,q+1}\delta_{r+1,s}\Big)\,. \qquad (A.7)
\end{aligned}
$$

Using (A.1),(A.2) and (A.5) we find

$$
\begin{aligned}
\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta &= \sum_{pq}\Big[\sqrt{p(p-1)}\;(-1)^{p+1}i^{p+q+1}\;\bar{\pi}^{p,q}\;\theta^{q,p-2}+\\
&\qquad +2\sqrt{p(q+1)}\;(-1)^{q+1}i^{p+q+1}\;\bar{\pi}^{p,q}\;\theta^{q+1,p-1}+\\
&\qquad +\sqrt{(q+1)(q+2)}\;(-1)^{p+1}i^{p+q+1}\;\bar{\pi}^{p,q}\;\theta^{q+2,p}\Big] \qquad (A.8)\\
\frac{1}{2}\bar{\pi}\tilde{\gamma}^{\oplus}\pi &= \sum_{pq}\sqrt{p}\;(-1)^{q+1}\;\bar{\pi}^{p,q}\;\pi^{q,p-1} \qquad (A.9)\\
\bar{\pi}\gamma^{\oplus}\gamma^{a}\theta &= i\sum_{pq}\Big[\sqrt{p}\;(-1)^{p}\;\bar{\pi}^{p,q}\;\gamma^{a}\;\theta^{q,p-1}+\sqrt{q+1}\;(-1)^{q+1}\;\bar{\pi}^{p,q}\;\gamma^{a}\;\theta^{q+1,p}\Big] \qquad (A.10)\\
R^{a} &= \frac{1}{4}\;\bar{\theta}\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}\gamma^{a}\theta\\
&= \frac{1}{2}\sum_{pqr}(-1)^{(q-p+r+1)r-qp-p}i^{q-p+1}\sqrt{\frac{p!(q-p+r)!}{q!r!}}\bar{\theta}^{p,q}\;\gamma^{a}\;\theta^{q-p+r,r}\times \qquad (A.11)\\
&\qquad \times[(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{r-p}+\Theta_{r-p-1})-(\Theta_{q-p-1}+\Theta_{q-p})(\Theta_{p-r-1}+\Theta_{p-r})]
\end{aligned}
$$

$$
\begin{aligned}
R^{\oplus} &= \frac{1}{4}\,\bar{\theta}\left\{\frac{1}{\tilde{\gamma}^{\oplus}},\gamma^{\oplus}\right\}\gamma^{\oplus}\theta \\
&= \frac{1}{2}\sum_{pqr}(-1)^{(q-p+r+2)r-qp-p}\sqrt{\frac{p!(q-p+r+1)!}{q!r!}}\bar{\theta}^{p,q}\;\theta^{q-p+r+1,r}\times \\
&\quad\times\left[(\Theta_{p-q}+\Theta_{p-q-1})(\Theta_{r-p+1}+2\Theta_{r-p}+\Theta_{r-p-1})-\right. \\
&\qquad\left.-(\Theta_{q-p-1}+\Theta_{q-p})(\Theta_{p-r-2}+2\Theta_{p-r-1}+\Theta_{p-r})\right].
\end{aligned} \tag{A.12}
$$

The component fields defined above satisfy

$$
\{\pi_{p,q},\theta^{r,s}] = \delta_p^r\delta_q^s\,. \tag{A.13}
$$

B. Subtle points in Sp(2) operators

In this appendix we explain some subtle points about $\frac{1}{\tilde{\gamma}^{\oplus}}$, due to the infinite dimensional structure of Sp(2) operators.

Consider the commutator

$$
\left\{\,\bar{\pi}\tilde{\gamma^{\oplus}}\theta\;,\;\left[\;\overset{(1)}{\theta}\overset{(0)}{\theta}\;,\;R^a\;\right]\;\right\}, \tag{B.1}
$$

where

$$
\begin{aligned}
\overset{(0)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}|\theta\rangle \\
&= \theta_0+\tilde{\theta} \\
&= \theta_0+\theta^{\oplus\ominus}+\theta^{\oplus\oplus\ominus\ominus}+\theta^{\oplus\oplus\oplus\ominus\ominus\ominus}+\cdots \\
\overset{(1)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})|\theta\rangle \\
&= 2i(\theta^{\oplus}\;-\;\sqrt{2}\theta^{\oplus\oplus\ominus}\;+\sqrt{3}\theta^{\oplus\oplus\oplus\ominus\ominus}\;-\;\sqrt{4}\theta^{\oplus\oplus\oplus\oplus\ominus\ominus\ominus}+\cdots)\,.
\end{aligned} \tag{B.2}
$$

We will also need:
for $n\geq 0$

$$
\begin{aligned}
\overset{(n)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n|\theta\rangle \\
&= \sum_{k=0}^{\infty}\;(-1)^{k(n+k+1)}i^{\frac{n(n+1)}{2}}\left(1+\Theta_{n-\frac{1}{2}}\right)\sqrt{\frac{(n+k)!}{k!}}\theta^{n+k,k}
\end{aligned} \tag{B.3}
$$

for $n<0$

$$
\begin{aligned}
\overset{(n)}{\theta} &= \langle 0|e^{ia_{\oplus}a_{\ominus}}(ia^{\dagger\oplus})^{|n|}|\theta\rangle \\
&= \sum_{k=0}^{\infty}\;(-1)^{k(|n|+k+1)}i^{\frac{|n|(|n|+1)}{2}}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{k!}{(|n|+k)!}}\theta^{k,|n|+k}\,.
\end{aligned} \tag{B.4}
$$

The above double commutator should vanish because the inner one involves only θ. However, if we apply the Jacobi identity we see

$$
\left\{\bar{\pi}\tilde{\gamma^{\oplus}}\pi,\left[\overset{(1)}{\theta}\overset{(0)}{\theta},R^a\right]\right\}=0=\left[\left\{\bar{\pi}\tilde{\gamma^{\oplus}}\pi,\,\overset{(1)}{\theta}\overset{(0)}{\theta}\right\},R^a\right]\;-\;\left\{\overset{(1)}{\theta}\overset{(0)}{\theta},[\bar{\pi}\tilde{\gamma^{\oplus}}\pi,R^a]\right\}
$$

$$
\begin{aligned}
&= [\tilde{0}, R^a] \;-\; 2\; \left\{ \overset{(1)}{\theta}\overset{(0)}{\theta}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta \right\} \\
&= [\tilde{0}, R^a] + 2\; \overset{(1)}{\theta}\,\gamma^a\, \overset{(1)}{\theta} + 4\; \overset{(0)}{\theta}\,\gamma^a\, \overset{(2)}{\theta}
\end{aligned}
\tag{B.5}
$$

where we have introduced the scalar $\tilde{0}$ defined by

$$
\begin{aligned}
\left[\frac{1}{2}\bar{\pi}\gamma^{\oplus}\pi, \overset{(n)}{\theta}\right] \quad (n \geq -1) \equiv & -i^{\frac{n(n+1)}{2}+n}\left(1+\Theta_{|n|-\frac{1}{2}}\right)\sqrt{\frac{(n+1)!}{0!}}(\bar{\pi}^{n+1,0}-\bar{\pi}^{n+1,0})+ \\
& +(-1)^n i^{\frac{n(n+1)}{2}+n}\left(1+\Theta_{|n|-\frac{1}{2}}\right)\sqrt{\frac{(n+2)!}{1!}}(\bar{\pi}^{n+2,1}-\bar{\pi}^{n+2,1})+ \\
& +\cdots+ \\
& +(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}\left(1+\Theta_{|n|-\frac{1}{2}}\right)\times \\
& \times\sqrt{\frac{(n+k+1)!}{k!}}(\bar{\pi}^{n+k+1,k}-\bar{\pi}^{n+k+1,k})+ \\
& +\cdots \\
\equiv\; & \tilde{0} \;\rightarrow\; 0
\end{aligned}
\tag{B.6}
$$

and

$$
\left[\overset{(n)}{\theta}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta\right] = -i^n\left(1+\Theta_{n-\frac{1}{2}}\right)\gamma^a \overset{(n+1)}{\theta} \; . \tag{B.7}
$$

Unfortunately, $\tilde{0}$ is not zero in the presence of R^a. Let's assume the collective $\overset{(n)}{\theta}_k$ has $k+1$ terms instead of an infinite number of terms. Then $[\frac{1}{2}\bar{\pi}\gamma^{\oplus}\pi, \overset{(n)}{\theta}_k]$ gives

$$
(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}\left(1+\Theta_{|n|-\frac{1}{2}}\right)\sqrt{\frac{(n+k+1)!}{k!}}\bar{\pi}^{n+k+1,k}
$$

Finally, one can see that

$$
(-1)^{(k+2)(n+k+2)-1} i^{\frac{n(n+1)}{2}+n}(1+\Theta_{|n|-\frac{1}{2}})\sqrt{\frac{(n+k+1)!}{k!}}[\bar{\pi}^{n+k+1,k}, R^a]
$$

gives $\;-i^n(1+\Theta_{|n|-\frac{1}{2}})\overset{(n+1)}{\theta}_k$. So if we make the collective $\overset{(n)}{\theta}_k$ have an infinite number of terms by $k \rightarrow \infty$ then $\tilde{0}$ produces a nonzero result in the commutator with R^a. This exactly cancels the remaining terms in (B.5) to make the double commutator consistent.

Keeping this subtle point in mind let's consider the following transformation

$$
\begin{aligned}
Q'_{\text{free}} &= e^{iR^a p_a}\tilde{Q}_{\text{free}}e^{-iR^a p_a} \\
&= e^{iR^a p_a}\Bigg(\frac{1}{2}c\Box \;-\frac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b+\tilde{0}b \;+\left[-iR^a p_a, -\frac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta+\tilde{0}\right]b+ \\
&\qquad +\frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi+\left[-iR^a p_a, \frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi\right]+\frac{1}{2}\left[-iR^a p_a, \left[-iR^a p_a, \frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi\right]\right]- \\
&\qquad -\frac{i}{2}\bar{\pi}\gamma^{\oplus}\not{p}\theta+\left[-iR^a p_a, -\frac{i}{2}\bar{\pi}\gamma^{\oplus}\not{p}\theta\right]\Bigg)e^{-iR^a p_a} \, ,
\end{aligned}
\tag{B.8}
$$

where $\tilde{0}b$ vanishes in Q'_{free} but will give a nontrivial contribution in the presence of R^a. We will determine this term from nilpotency of Q'_{free}.

In terms of $R^\oplus$ and $\overset{(n)}{\theta}$, $\tilde{Q}_{free}$ is

$$
\begin{aligned}
\tilde{Q}_{\rm free} = {} & \frac{1}{2}c\Box \; - \frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta b - \overset{(2)}{\theta}\,\gamma^a\,\overset{(0)}{\theta}\,p_a b + \frac{1}{2}\,\overset{(1)}{\theta}\,\gamma^a\,\overset{(1)}{\theta}\,p_a b + \\
& + \tilde{0}b + [-iR^a, \tilde{0}]p_a b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi \; - \; \tilde{0}^a p_a + \\
& + \frac{1}{2}[-iR^a p_a, -\tilde{0}^b p_b] + \frac{1}{2}R^\oplus\Box\,,
\end{aligned} \tag{B.9}
$$

where

$$
\left[-iR^a p_a, -\frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta\right] b = -\,\overset{(2)}{\theta}\,\gamma^a\,\overset{(0)}{\theta}\,p_a b + \frac{1}{2}\,\overset{(1)}{\theta}\,\gamma^a\,\overset{(1)}{\theta}\,p_a b \tag{B.10}
$$

and we have defined the vector $\tilde{0}^a$

$$
\tilde{0}^a \equiv -\left[-iR^a, \frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi\right] + \frac{i}{2}\bar{\pi}\gamma^\oplus\gamma^a\theta \tag{B.11}
$$

The nilpotency of Q'_{free} implies

$$
-\left\{-\tilde{0}^a, \frac{1}{2}R^\oplus\right\} p_a\Box = \frac{1}{2}\left(-\,\overset{(2)}{\theta}\,\gamma^a\,\overset{(0)}{\theta}\, + \frac{1}{2}\,\overset{(1)}{\theta}\,\gamma^a\,\overset{(1)}{\theta}\, + [-iR^a, \tilde{0}]\right) p_a\Box \tag{B.12}
$$

$$
\begin{aligned}
\left[\frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta, \frac{1}{2}[-iR^a p_a, -\tilde{0}^b p_b]\right] \; b &= -\left[-\frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta + \tilde{0}, \frac{1}{2}R^\oplus\right]\Box\, b \\
&= \frac{1}{4}\left(3i\,\overset{(1)}{\theta}\,\overset{(2)}{\theta}\, - 2i\,\overset{(0)}{\theta}\,\overset{(3)}{\theta}\, - 2[\tilde{0}, R^\oplus]\right)\Box\, b\,.
\end{aligned} \tag{B.13}
$$

From (B.11) we can find $\tilde{0}^a$ as

$$
\tilde{0}^a = \frac{i}{4}\,\overset{(1)}{\pi}_\infty\gamma^a\,\overset{(0)}{\theta}\, - \frac{i}{4}\,\overset{(0)}{\pi}_\infty\,\gamma^a\,\overset{(1)}{\theta}\,, \tag{B.14}
$$

where

$$
\overset{(n)}{\chi}_\infty \equiv \lim_{k\to\infty}(-1)^{k(n+k+1)}(-i)^{\frac{n(n+1)}{2}}\sqrt{\frac{(n+k+1)!}{k!}}\chi^{n+k,k} \tag{B.15}
$$

for $\chi = \theta, \pi$. Inserting this into (B.12) we find

$$
[iR^a, \tilde{0}] = \overset{(1)}{\theta}\,\gamma^a\,\overset{(1)}{\theta} \tag{B.16}
$$

$$
[\tilde{0}, R^\oplus] = 2i\,\overset{(1)}{\theta}\,\overset{(2)}{\theta}\;. \tag{B.17}
$$

One solution for $\tilde{0}$ is

$$
\tilde{0} = \overset{(1)}{\pi}_\infty\,\overset{(1)}{\theta}\;. \tag{B.18}
$$

Now $\tilde{Q}_{free}$ is nilpotent, as it should be. However, the origin of collective nonminimal fields is that R^a produces $\bar{\pi}\gamma^\oplus\gamma^a\theta$ using $[\bar{\pi}\tilde{\gamma}^\oplus\pi, \overset{(n)}{\theta}] \to 0$. (The key feature of R^a is $\overset{(n)}{\theta}$, see (A.11).) And this implies that R^a and $R^\oplus$ commute with $\bar{\pi}\gamma^\oplus\gamma^\oplus\theta$ only up to these collective nonminimal fields (see (A.6)). So these collective fields are just mathematical objects to compensate terms like $[R^i, \tilde{0}]$. Therefore, a physical (but not mathematical) equivalent is to drop $\tilde{0}^{(a)}$ and $\overset{(n)}{\theta}$ and regard R^i as terms commuting with $\bar{\pi}\gamma^\oplus\gamma^\oplus\theta$ in (B.9).

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The resulting $\tilde{Q}_{free}$ is simply

$$\tilde{Q}_{free} = \frac{1}{2}c\Box \; - \frac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi + \frac{1}{2}R^{\oplus}\Box \tag{B.19}$$

with $[\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta, R^{\oplus}] \sim 0$.

Our results for $\overset{\circ}{Q}{}'_{YMB}$ (3.10), $\overset{\circ}{Q}_{SYMB}$ (3.29) and Q''_{SYMB} (3.40) all reflect this prescription. $\tilde{\theta}$ in these BRST operators will produce only $\pi^{\oplus}$, $\theta^{\oplus}$ and $\theta^{\oplus\oplus}b$ ($\theta^{\oplus\oplus}b$ will only appear in (3.10)) dropping $\overset{(1)}{\theta}$ and $\overset{(2)}{\theta}\, b$. If we want to be rigorous $\overset{(n)}{\theta}$ and $\tilde{0}^{(a)}$ should be kept and the commutator $[R^i, \bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta]$ should be calculated for both to compensate terms from $\tilde{0}^{(a)}$.

If we consider this mathematical rigor for Q'_{YMB}(3.10) we find

$$\begin{aligned} Q^{\text{collective}}_{YMB} = e^{iR^a\nabla_a}\Bigg[& \frac{1}{2}c(\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) + \frac{1}{2}R^{\oplus}(\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) + \\ & + \frac{1}{2D}[iR^a, \tilde{0}_a](\Box - \bar{\pi}\gamma^{ab}\theta F_{ab}) \; - \frac{1}{2}\bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^{\oplus}\pi + \\ & + \tilde{0}b \; - \; \overset{(2)}{\theta}\,\gamma^a\,\overset{(0)}{\theta}\,\nabla_a b + \frac{1}{2}\,\overset{(1)}{\theta}\,\gamma^a\,\overset{(1)}{\theta}\,\nabla_a b \; - \; [iR^a, \tilde{0}]\nabla_a b - \\ & - \tilde{0}^a\nabla_a - \\ & - \frac{1}{2}\left\{(cb - \frac{1}{2}) + R^{\oplus}b + \frac{1}{2D}[iR^a, \tilde{0}_a]b\right\} \times \\ & \times \; \bar{\theta}\frac{1}{\tilde{\gamma}^{\oplus}}\gamma^{ab}\theta[F_{ab}, \nabla_c]\left(-\;\overset{(2)}{\theta}\,\gamma^c\,\overset{(0)}{\theta} + \frac{1}{2}\,\overset{(1)}{\theta}\,\gamma^c\,\overset{(1)}{\theta} \; - \; [iR^c, \tilde{0}]\right) + \\ & + \frac{1}{2}\left\{c + R^{\oplus} + \frac{1}{2D}[iR^a, \tilde{0}_a]\right\} \times \\ & \times \; \bar{\theta}\frac{1}{\tilde{\gamma}^{\oplus}}\gamma^{ab}\theta[F_{ab}, \nabla_c]\tilde{0}^c \;\Bigg]\; e^{-iR^a\nabla_a} \;\Big|_{\text{ linear in } F,\ [\nabla, F]=0}\,, \end{aligned} \tag{B.20}$$

where

$$\begin{aligned} D &= \text{dimension of space-time} \qquad \text{(10 here)} \\ \text{and} \quad \Box &= -\nabla^a\nabla_a\,. \end{aligned} \tag{B.21}$$

If we drop $\overset{(n)}{\theta}$ and $\tilde{0}^{(a)}$ and regard $[\gamma^{\oplus}, \frac{1}{\tilde{\gamma}^{\oplus}}] = 0$ (which is equivalent to $[R^i, \bar{\pi}\gamma^{\oplus}\gamma^{\oplus}\theta]] = 0$) we come back to Q'_{YMB} (3.10). The last four lines do not contribute to Q'_{YMB} but are there for nonconstant Yang-Mills background.

C. Regularization

In this appendix we will consider a regularization procedure which will give the prescription of the previous appendix. The motivation is the fact that $[\bar{\pi}\tilde{\gamma}^{\oplus}\pi, \overset{(n)}{\theta}\,] \to 0$. However, this does not exactly vanish, but leaves a piece of $\overset{(n+1)}{\pi}{}_{\infty}$. This remnant gives a nontrivial contribution in the presence of $\frac{1}{\tilde{\gamma}^{\oplus}}$, i.e.,

$$\langle 0|e^{ia_{\oplus}a_{\ominus}}(ia_{\ominus})^n\tilde{\gamma}^{\oplus}|\theta\rangle \to 0$$

$$\langle 0|e^{ia_\oplus a_\ominus}(ia_\ominus)^n \tilde{\gamma}^\oplus \frac{1}{\tilde{\gamma}^\oplus}|\theta\rangle \rightarrow \overset{(n)}{\theta} \,. \tag{C.1}$$

Now if we introduce some regularization parameter z as

$$\langle p,q|\tilde{\gamma}^\oplus|r,s\rangle \overset{\text{regularized}}{\longrightarrow} z^{p+r+s+r}\langle p,q|\tilde{\gamma}^\oplus|r,s\rangle \,, \qquad z\rightarrow 1 \tag{C.2}$$

then (C.1) becomes

$$\begin{aligned} \langle 0|e^{ia_\oplus a_\ominus}(ia_\ominus)^n \tilde{\gamma}^\oplus|\theta\rangle_k &\rightarrow z^{2n+2k+1}\ \theta^{n+k+1,k} \\ \langle 0|e^{ia_\oplus a_\ominus}(ia_\ominus)^n \tilde{\gamma}^\oplus \frac{1}{\tilde{\gamma}^\oplus}|\theta\rangle_k &\rightarrow z^{2n+2k+1}\ \overset{(n)}{\theta}_k \end{aligned} \tag{C.3}$$

where k means the collective field has $k+1$ terms. The regularization is to send k to infinity with fixed z (<1). Then the operator $\tilde{\gamma}^\oplus \frac{1}{\tilde{\gamma}^\oplus}$ is a projection operator which will project out the collective fields. The free terms in any $Q_{backgroud}$ all have this projection operator which goes to the identity in the absence of $\frac{1}{\tilde{\gamma}^\oplus}$.

$$\begin{aligned} &-\frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\theta b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\pi\ -\ i\frac{1}{2}\nabla_a\bar{\pi}\gamma^\oplus\gamma^a\theta \rightarrow \\ &\qquad \rightarrow -\frac{1}{2}\bar{\pi}\gamma^\oplus\gamma^\oplus\Pi\theta b + \frac{1}{4}\bar{\pi}\tilde{\gamma}^\oplus\Pi\pi\ - i\frac{1}{2}\nabla_a\bar{\pi}\gamma^\oplus\gamma^a\Pi\theta \,. \end{aligned} \tag{C.4}$$

The arrow means inserting the projection operator and dropping collective fields after expansion of exponential factors.

If a collective field is truncated, i.e., with incomplete beginning or ending components, then we cannot remove it by this regularization procedure, but we can still avoid its contribution. This situation occurs when we consider the commutator

$$\{\eta\pi^{p,p+1},[\nabla_0\tilde{\theta},\bar{\pi}\gamma^\oplus\gamma^a\theta]\} = \{\eta\pi^{p,p+1},-i\nabla_0\gamma^a\theta^\oplus\} = 0 \qquad (p>0)\,, \tag{C.5}$$

where η is a constant fermionic field. This implies

$$[\nabla_0\tilde{\theta},(\pi^{p,p}+\pi^{p+1,p+1})\eta] = 0 \,. \tag{C.6}$$

The second argument in the commutator is an example of a truncated collective field. Actually, this commutator indeed vanishes if we consider $\overset{(1)}{\theta}$, which we drop in regularization. So for consistent regularization we take this as vanishing.

This fact is applied for closure of the algebra

$$[\eta\theta^{q,q+1},\{\ [\eta'\pi^{p,p+1},Q_R],Q_R\}] \tag{C.7}$$

where Q_R is any version of the BRST operator including R^i. After canceling ghost-number-nonzero components this commutator reduces to

$$[\eta' C_p^-,\eta C_q^+]\,, \tag{C.8}$$

where $C_p^\pm$ will be superstring constraints if we use $Q_{sstring}$. But the original commutator is just

$$\frac{1}{2}[\eta\theta^{q,q+1},\ [\eta'\pi^{p,p+1},\{Q_R,Q_R\}]] \tag{C.9}$$

and it is just zero due to nilpotency of Q_R. However, $\eta' C_p^-$ has a term like $\eta\gamma^a(\pi^{p,p} + \pi^{p+1,p+1})$, which comes from $[\eta'\pi^{p,p+1}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta]$. But this combination of $\pi^{p,p}$ is just an example of truncated collective fields. This truncated collective field will interact with $\tilde{\theta}$ in ηC_q^+ giving a nonzero contribution in this apparently vanishing commutator. This should be canceled by a $[\eta'\pi^{p,p+1}, \overset{(1)}{\theta}]$ contribution, which we projected out by regularization. What this means is that for consistent regularization we should take the commutator with truncated collective fields, $\tilde{\theta}$, R^i as vanishing. For example, we should take $[(\pi^{p,p}+\pi^{p+1,p+1}), \tilde{\theta}]$ as zero but we should calculate $\{(\theta^{p,p} + \theta^{p+1,p+1}), (\pi^{p,p} + \pi^{p+1,p+1})\}$ in $[\eta' C_p^-, \eta C_p^+]$, both of which come from $\bar{\pi}\gamma^{\oplus}\gamma^a\theta$.

D. Closure of constraints

First of all, if one directly calculates $[\pi^{p,p+1}, R^{\oplus}]$ one gets

$$\begin{aligned}
[\pi^{p,p+1}, R^{\oplus}] &= -\frac{3}{4}\sum_r(-1)^p\sqrt{p+1}\theta^{r,r}(\Theta_{r-p} + 2\Theta_{r-p-1} + \Theta_{r-p-2}) + \\
&\quad + \frac{1}{4}\sum_r(-1)^p\sqrt{p+1}\theta^{r,r}(\Theta_{p-r+1} + 2\Theta_{p-r} + \Theta_{p-r-1}) \\
&\equiv -\frac{3}{4}A + \frac{1}{4}B \\
&= -\frac{1}{2}(A-B) - \frac{1}{4}(A+B)\,.
\end{aligned} \tag{D.1}$$

But $A+B$ is just $(-1)^p\sqrt{p+1}\ \overset{(0)}{\theta}$, and it will vanish when it acts on the projection operator Π. Also,

$$A - B \ = \ -2(-1)^p\sqrt{p+1}\vartheta_p$$

in terms of ϑ (4.17). When we calculate closure of the constraints, there are two types of terms related to the above expression, i.e.,

$$\{[\eta\pi^{p,p+1}, R^{\oplus}], [\eta'\theta^{q,q+1}, \bar{\pi}\tilde{\gamma}^{\oplus}\pi]\}$$

and

$$\begin{aligned}
&\{[\eta\pi^{p,p+1}, R^{\oplus}], [\eta'\pi^{q,q+1}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta]\} \\
&\{[\eta\pi^{p,p+1}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta], [\eta'\pi^{q,q+1}, R^{\oplus}]\}\,,
\end{aligned}$$

where η and η' are constant spinors.

The first type is always zero due to the symmetry of $\gamma^{\oplus}$ and $\tilde{\gamma}^{\oplus}$. If $p \neq q$, $p \neq q+1$ and $p+1 \neq q$ the second type cancels because of the $A - B$ sign difference. If $p = q$ we can express $[\eta\pi^{p,p+1}, R^{\oplus}]$ as just $-\sqrt{p+1}\eta(\theta^{p,p} - \theta^{p+1,p+1})$. If $p = q+1$ or $p+1 = q$ we can use $\sqrt{p+1}\eta(\theta^{p,p} - \theta^{p+1,p+1})$. This becomes clearer in a simpler situation,

$$\{R^{\oplus}, \bar{\pi}\gamma^{\oplus}\gamma^a\theta\} = 0\,.$$

Then we have

$$\{\eta'\pi^{q,q+1}, [\eta\pi^{p,p+1}, \{R^{\oplus}, \bar\pi\gamma^{\oplus}\gamma^a\theta\}]\} = 0\,.$$

But this implies

$$[\{\eta\pi^{p,p+1}, R^{\oplus}\}, \{\eta'\pi^{q,q+1}, \bar\pi\gamma^{\oplus}\gamma^a\theta\}] + [\{\eta'\pi^{q,q+1}, R^{\oplus}\}, \{\eta\pi^{p,p+1}, \bar\pi\gamma^{\oplus}\gamma^a\theta\}] = 0\,.$$

This is one part of the closure of constraints. A similar analysis shows

$$\begin{aligned}
&\Big[\ \eta\hat P_a\gamma^a(\theta^{p,p}+\theta^{p+1,p+1}),\ \eta'\hat P_a\gamma^a(\theta^{p,p}+\theta^{p+1,p+1})\ \Big] \\
&\quad\sim \left[\ \eta(\pi^{p,p}-\pi^{p+1,p+1}),\ \frac{i}{4}\tilde R_a\ \right]\ \eta'\gamma^a(\theta^{p,p}+\theta^{p+1,p+1}) + \\
&\qquad +\ \eta\gamma^a(\theta^{p,p}+\theta^{p+1,p+1})\left[\ \frac{i}{4}\tilde R_a, \eta'(\pi^{p,p}-\pi^{p+1,p+1})\right],
\end{aligned} \tag{D.2}$$

where $\sim$ means we should drop truncated collective fields.

Secondly, from the fact that $[\tilde\theta, \bar\pi\gamma^{\oplus}\gamma^a\theta] = \overset{(1)}{\theta}\ -i\,\theta^{\oplus} \sim -i\,\theta^{\oplus}$ due to regularization when $\tilde\theta$ acts on terms from $\bar\pi\gamma^{\oplus}\gamma^a\theta$, we should change the sign of $\tilde\theta$. (More precisely, they are all zero except for $\pi^{1,1}$ from $\bar\pi\gamma^{\oplus}\gamma^a\theta$. For only this term we can see this sign-change effect as explained in the previous appendix.) This fact was implicitly expressed with the projection operator Π in the constraints.

Finally, one should be cautious about $\{\bar\pi\gamma^{\oplus}\gamma^a\theta\ ,\ \bar\pi\gamma^{\oplus}\gamma^b\theta\} = (-1)(-2)g^{ab}\ \bar\pi\gamma^{\oplus}\gamma^{\oplus}\theta$. The "$-$" sign comes from the fact that OSp(2) gamma matrices (and therefore $a^\dagger$ and a) anticommute with ordinary gamma matrices. This gives an additional sign when one calculates terms like $[\eta\gamma^a\theta^{p,p}\ ,\ \eta'\gamma^b\pi^{p,p}]$.

E. ZJBV form of BRST

The ZJBV form of the BRST operator follows from the hamiltonian form of the BRST operator

$$Q^{\mathrm{ZJBV}} = \frac{i}{2}\dot\phi^A\phi_A - \check\phi_A\{\phi^A, Q^H]\tag{E.1}$$

Then in our case,

$$Q^{\mathrm{ZJBV}}_{\mathrm{sstring}} = -\dot X\cdot P^0\ + \sum_{\pm} Q^{(\pm)\mathrm{ZJBV}}_{\mathrm{sstring}}\tag{E.2}$$

where, e.g., for the $(+)$ term (for $(-)$, just add a $-$ for each $'$)

$$\begin{aligned}
Q^{\mathrm{ZJBV}}_{\mathrm{sstring}} &= \dot c b + \dot\theta\pi + \\
&\quad + (\check X_a - \check P'_a)\Bigg[\left(c + \frac{1}{2}\bar\theta\tilde\gamma^{\oplus}\theta\right)\hat P^a\ -\ \frac{i}{2}\bar\pi\gamma^{\oplus}\gamma^a\theta - \\
&\qquad - \left(i\theta_0\gamma_a\theta'_0 + 2i\tilde\theta\gamma_a\theta'_0 + i\tilde\theta\gamma_a\tilde\theta'\ -\frac{1}{2}\ R'_a|_{>}\right)R^{\oplus} +\ i\frac{1}{2}\bar\theta^{\oplus}\theta_0\ \theta_0\gamma^a\theta'_0 - \\
&\qquad - \frac{1}{2}\bar\theta^{\oplus}\gamma^b\gamma^a\theta_0\left(i\frac{1}{2}\theta_0\gamma_b\theta'_0 + 2i\tilde\theta\gamma_b\theta'_0 + i\tilde\theta\gamma_b\tilde\theta' - \frac{1}{2}\ R'_a|_{>}\right) -
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - \frac{1}{3}\bar{\theta}^{\oplus}\gamma^a\gamma^b\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta_0' + i\frac{5}{4}\ \tilde{\theta}\gamma_a\tilde{\theta}' - \frac{3}{4}\ R_a'|_>\right)\Bigg] - \\
&- \check{c}\left[icc' + 2\ \bar{\pi}a^{\dagger\oplus}a^{\oplus}\theta|_>\right] - \\
&- \check{b}\left(\ -\frac{1}{2}\hat{P}^2\ - 2ic'b\ - icb'\ -\ i\bar{\theta}'\pi\ -\ i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]'\right) + \\
&+ \sum_{p>1}\ \check{\theta}_{p,p+1}\Bigg[\left\{i\left(c + R^{\oplus} + \frac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta\ \right)|_>\theta'^{p,p+1}\right\} - \\
&\qquad\qquad - 2i[(p+1) - \sqrt{p+1}\frac{\sqrt{p}+\sqrt{p+2}}{2}]\Pi\theta^{p+1,p}b + \\
&\qquad\qquad + (-1)^{p+1}i\frac{1}{2}\sqrt{p+1}\Pi(\pi^{p,p} - \pi^{p+1,p+1}) + \\
&\qquad\qquad + (-1)^{p+1}i\frac{1}{2}\sqrt{p+1}\gamma^a\Pi(\theta^{p,p} + \theta^{p+1,p+1})\mathcal{P}_a\Bigg] - \\
&- \frac{i}{2}\ \check{\theta}_{0,1}\Bigg\{\pi_0 + (\gamma^a\theta_0)\hat{P}_a + i\frac{1}{2}(\gamma^a\theta_0)\theta_0\gamma_a\theta_0' + \\
&\qquad\quad + \frac{2}{3}\gamma^a\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta_0' + i\frac{5}{4}\ \tilde{\theta}\gamma_a\tilde{\theta}' - \frac{3}{4}\ R_a'|_>\right) - \pi^{1,1} + \gamma^a\Pi\theta^{1,1}\mathcal{P}_a\Bigg\} - \\
&- \sum_{p>1}\ \check{\pi}_{p,p+1}\Bigg[\left\{i\left(c + R^{\oplus} + \frac{1}{2}\bar{\theta}\tilde{\gamma}^{\oplus}\theta\ \right)|_>\pi^{p,p+1}\right\}' + \\
&\qquad\qquad + (-1)^{p+1}i\sqrt{p+1}\vartheta^p \times \\
&\qquad\qquad \times \left\{-\frac{1}{2}\mathcal{P}^2 + \frac{1}{2}\hat{P}^2 + i\bar{\theta}'\pi + i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]'\right\} + \\
&\qquad\qquad + \frac{1}{2}R^{\oplus}|_>\sum_r(-1)^{r^2+p+1}\sqrt{\frac{p+1}{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-p-1} + \Theta_{r-p-2})\mathcal{P}_a - \\
&\qquad\qquad -\ (-1)^{p+1}i\sqrt{p+1}(\theta^{p,p} - \theta^{p+1,p+1}) \times \\
&\qquad\qquad \times \left(\ \frac{1}{2}\hat{P}^2 + i\bar{\theta}'\pi + i[\bar{\theta}(a^{\dagger\oplus}a_{\oplus} - a^{\dagger\ominus}a_{\ominus})\pi]'\right) + \\
&\qquad\qquad +\ 2i\left[(p+1) - \sqrt{p+1}\frac{\sqrt{p}+\sqrt{p+2}}{2}\right]\pi^{p+1,p}b - \\
&\qquad\qquad - (-1)^{p+1}i\frac{1}{2}\sqrt{p+1}\gamma^a\Pi(\pi^{p,p} + \pi^{p+1,p+1})\mathcal{P}_a + \\
&\qquad\qquad + \frac{1}{4}\sum_r(-1)^{r^2+p+1}\sqrt{\frac{p+1}{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-p-1} + \Theta_{r-p-2}) \times \\
&\qquad\qquad \times \left(-\bar{\pi}\gamma^{\oplus}\gamma^a\theta|_> + i\bar{\theta}^{\oplus}\gamma^a \times \right. \\
&\qquad\qquad\quad \left. \times \left(\pi_0 + (\gamma^b\theta_0)\hat{P}_b + i\frac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta_0'\ \right) + i\bar{q}^{\oplus}\gamma_a\tilde{\theta}\right)\Bigg] - \\
&- \frac{i}{2}\ \check{\pi}_{0,1}\left\{\gamma^a\left(\pi_0 + (\gamma^b\theta_0)\hat{P}_b + i\frac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta_0'\ \right) \times \right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\hat{P}_a\ +i\theta_0\gamma_a\theta_0'\ +2i\tilde{\theta}\gamma_a\theta_0'+i\tilde{\theta}\gamma_a\tilde{\theta}'-\frac{1}{2}\ R_a'|_>\ \right)+\\
&+\frac{2}{3}\gamma^b\gamma^a\tilde{\theta}\left(2i\tilde{\theta}\gamma_a\theta_0'+i\frac{5}{4}\ \tilde{\theta}\gamma_a\tilde{\theta}'-\frac{3}{4}\ R_a'|_>\right)\times\\
&\times\left(\hat{P}_b\ +i\theta_0\gamma_b\theta_0'\ +2i\tilde{\theta}\gamma_b\theta_0'+i\tilde{\theta}\gamma_b\tilde{\theta}'-\frac{1}{2}\ R_b'|_>\ \right)-\\
&-\gamma^a\Pi\pi^{1,1}\left(\hat{P}_a\ +i\theta_0\gamma_a\theta_0'\ +2i\tilde{\theta}\gamma_a\theta_0'+i\tilde{\theta}\gamma_a\tilde{\theta}'-\frac{1}{2}\ R_a'|_>\ \right)+\\
&+2\theta^{1,1}\left(\ -\frac{1}{2}\hat{P}^2\ -\ i\bar{\theta}'\pi\ -\ i[\bar{\theta}(a^{\dagger\oplus}a_\oplus-a^{\dagger\ominus}a_\ominus)\pi]'\right)+\\
&+2\vartheta^0\times\left\{-\frac{1}{2}\left(\hat{P}_a\ +i\theta_0\gamma_a\theta_0'\ +2i\tilde{\theta}\gamma_a\theta_0'+i\tilde{\theta}\gamma_a\tilde{\theta}'\ -\frac{1}{2}\ R_a'|_>\right)^2+\right.\\
&\qquad\left.+\frac{1}{2}\hat{P}^2+i\bar{\theta}'\pi+i[\bar{\theta}(a^{\dagger\oplus}a_\oplus-a^{\dagger\ominus}a_\ominus)\pi]'\right\}+\\
&+\frac{1}{2}R^\oplus|_>\sum_r(-1)^{r^2+1}\frac{1}{\sqrt{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-1}+\Theta_{r-2})\mathcal{P}_a-\\
&-\frac{i}{4}\sum_r(-1)^{r^2+1}\frac{1}{\sqrt{r}}\gamma^a\theta'^{r-1,r}(\Theta_{r-1}+\Theta_{r-2})\times\\
&\times\Bigg(-\bar{\pi}\gamma^\oplus\gamma^a\theta|_>+i\bar{\theta}^\oplus\gamma^a\times\\
&\quad\times\left(\pi_0+(\gamma^b\theta_0)\hat{P}_b+i\frac{1}{2}(\gamma^b\theta_0)\theta_0\gamma_b\theta_0'\ \right)+i\bar{q}^\oplus\gamma_a\tilde{\theta}\Bigg)\Bigg\}-\\
-&\sum_{q\neq p+1}\breve{\theta}_{p,q}[\theta^{p,q},Q_{\text{sstring}}\}-\\
-&\sum_{q\neq p+1}\breve{\pi}_{p,q}[\pi^{p,q},Q_{\text{sstring}}\}\,,
\end{aligned}
$$

where

$$
\Theta_x=\begin{cases}1 & x\geq 0\\ 0 & x<0\,.\end{cases}\tag{E.3}
$$

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